

Filtering with Randomised Markov Bridges

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Abstract

We consider the filtering problem of estimating a hidden random variable X by noisy observations. The noisy observation process is constructed by a randomised Markov bridge (RMB) $(Z_t)_{t \in [0, T]}$ of which terminal value is set to $Z_T = X$. That is, at the terminal time T , the noise of the bridge process vanishes and the hidden random variable X is revealed. We derive the explicit filtering formula, also called the Bayesian posterior probability formula, for a general RMB. It turns out that the posterior probability is given by a function of the current time t , the current observation Z_t , the initial observation Z_0 , and the prior distribution ν of X .

Keywords: Randomized Markov bridge; hidden random variable; filtering; Bayesian estimation.

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1 Introduction

Let $E \subset \mathbb{R}^n$ and let $T \in (0, \infty)$. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider an E -valued (y, T, z) -Markov bridge $Y^{(y, T, z)} := (Y_t^{(y, T, z)})_{t \in [0, T]}$ so that

$$Y_0^{(y, T, z)} = y, \quad Y_T^{(y, T, z)} = z \quad \text{a.s.}, \quad (1.1)$$

and an E -valued random variable X , which is independent of $Y^{(y, T, z)}$. Here, by a (y, T, z) -Markov bridge, we mean a process obtained by conditioning a Markov process $Y := (Y_t)_{t \geq 0}$ to start in $y \in E$ at time 0 and arrive at $z \in E$ at time $T \in (0, \infty)$. For its construction, we follow Fitzsimmons et al. (1993). We define the process $Z := (Z_t)_{t \in [0, T]}$ by

$$Z := Y^{(y, T, X)},$$

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which we call *randomised Markov bridge* (RMB). We further define

$$\mathcal{F}_t^Z := \sigma(Z_s; s \in [0, t]), \quad t \in [0, T],$$

and let Z be the noisy observation process of the hidden random variable X . At the terminal time T , the noise in the bridge process $Y^{(y, T, z)}$ vanishes and the hidden variable $X (= Z_T)$ is revealed. We are interested in the stochastic filtering (or Bayesian estimation) problem for the hidden random variable X through the observation of Z . That is, we are interested in computing the conditional probability,

$$\pi_t(dx) := \mathbb{P}(X \in dx | \mathcal{F}_t^Z), \quad t \in [0, T], \quad (1.2)$$

and the conditional expectation,

$$\pi_t(f) := \mathbb{E}[f(X) | \mathcal{F}_t^Z] = \int_E f(x) \pi_t(dx), \quad t \in [0, T], \quad (1.3)$$

where $f : E \rightarrow \mathbb{R}$ is Borel-measurable so that $f(X)$ is integrable.

The problem (1.2)-(1.3) of filtering a hidden random variable X by observing an RMB process Z has been studied in a financial context by, e.g., Brody *et al.* (2008), Hoyle *et al.* (2011), and Filipović *et al.* (2012). In their financial applications, $f(X)$ represents the cash flow of a financial asset that is paid at the terminal date T , and Z is called the information process. The conditional expectation (1.3) is used to model the asset price process $(S_t)_{t \in [0, T]}$ by

$$S_t := e^{-r(T-t)} \pi_t(f) = \mathbb{E} \left[e^{-r(T-t)} f(X) \mid \mathcal{F}_t^Z \right],$$

where $r(\geq 0)$ is the risk-free interest rate and \mathbb{P} is regarded as being the so-called risk-neutral probability measure. In the above-mentioned works, Brownian random bridges and, more generally, Lévy random bridges are employed, and the stochastic dynamics of asset prices are derived and computed.

The main goal of the present article is to provide the explicit representations of (1.2) and (1.3) for a *general* RMB, focusing on its interesting features from a stochastic filtering viewpoint. It is well-known that in the general stochastic filtering problem, the conditional probability (1.2) has an infinite-dimensional structure; it is the solution to the measure-valued Kushner-Stratonovich stochastic differential equation (SDE). In the problem we consider, unlike in the general situation, we obtain the relation

$$\mathbb{P}(Z_t \in dy | \mathcal{F}_s^Z) = \mathbb{P}(Z_t \in dy | Z_s, Z_0) =: P_{s,t}(Z_s, dy | Z_0) \quad (1.4)$$

for $0 \leq s < t \leq T$ (see Propositions 2.1 and 2.2, and Remark 2.2 for details). In words, we can regard Z as satisfying the Markov property with respect to its natural filtration $(\mathcal{F}_t^Z)_{t \in [0, T]}$, once the initial value Z_0 is fixed (see Proposition 2.2). From the above relation, it follows that

$$\pi_t(dz) = \mathbb{P}(Z_T \in dy | \mathcal{F}_t^Z) = P_{t,T}(Z_t, dy | Z_0).$$

We thus observe that the pair of observations (Z_0, Z_t) is a “sufficient statistic” to describe $\pi_t(dz)$, and the past observation $(Z_s)_{s \in (0,t)}$ is not necessary for the computation of the Bayesian posterior probability. As a consequence, the dynamics of $(\pi_t(dz))_{t \in [0,T]}$ can be determined by a finite-dimensional Markovian SDE (see Proposition 2.3).

Remark 1.1. We refer to Baudoin (2002) for a closely related piece of work: In the Wiener space setup, Baudoin (2002) introduces conditioned stochastic differential equations (CSDE), where the solution to the SDE is conditioned by (the law of) a random variable. If the solution to a Markovian SDE is conditioned by some given terminal law, the associated CSDE is nothing but an RMB as considered in this article. Although the work by Baudoin (2002) and the analysis presented in this article overlap in places, the following viewpoint and motivation appear to be different. For example, (a) we are interested in providing a stochastic filtering interpretation of RMBs, and (b) we consider general RMBs whereas in Baudoin (2002) develops Brownian CSDEs.

In the next section, after preparing the setup in detail, we state our results and give additional explanations in the remarks. The proofs are collected in Section 3.

2 Results

Let $E \subset \mathbb{R}^n$ be a Borel state space, and let $T \in (0, \infty)$ be a given constant. For the construction of the E -valued (y, T, z) -Markov bridge $Y^{(y,T,z)} := (Y_t^{(y,T,z)})_{t \in [0,T]}$ which satisfies (1.1), we follow Section 2 of Fitzsimmons et al. (1993). We consider an E -valued strong Markov process $Y := (Y_t)_{t \in [0,T]}$ with càdlàg sample paths, which is realized as the coordinate process Y on Ω_T^1 , that is, on the space of right continuous paths from $[0, T)$ to E that have left limits on $(0, T)$. The law of the Markov process Y that starts from y is denoted by $\tilde{\mathbb{P}}_y^1$, and the natural filtration of Y is denoted by $(\mathcal{F}_t^1)_{t \in [0,T]}$. We assume that the transition probability of the Markov process Y has the density

$$\tilde{P}_t(x, dy) = \tilde{p}_t(x, y)m(dy)$$

with respect to a σ -finite measure m on E such that the Chapman-Kolmogorov identity

$$\tilde{p}_{t+s}(x, z) = \int_E \tilde{p}_t(x, y)\tilde{p}_s(y, z)m(dy)$$

holds true, where $t > 0$ and $s > 0$ ($s+t \leq T$), $x \in E$ and $z \in E$, and $\tilde{p}_t(x, y) > 0$ for all $(t, x, y) \in (0, T] \times E \times E$. By Proposition 1 of Fitzsimmons et al. (1993), we see the following:

- (a) Let $\mathcal{F}_{T-}^1 := \sigma(\cup_{0 \leq t < T} \mathcal{F}_t^1)$. For each y and $z \in E$, we can construct the probability measure $\mathbb{P}_{y,z}^1$ on $(\Omega_T^1, \mathcal{F}_{T-}^1)$ that satisfies

$$d\mathbb{P}_{y,z}^1 \mid_{\mathcal{F}_t^1} = \Lambda_t^{(y,T,z)} d\tilde{\mathbb{P}}_y^1 \mid_{\mathcal{F}_t^1}$$

for all $t \in [0, T)$, where

$$\Lambda_t^{(y, T, z)} := \frac{\tilde{p}_{T-t}(Y_t, z)}{\tilde{p}_T(y, z)}.$$

- (b) The law of $(Y_t)_{t \in [0, T]}$ under $\mathbb{P}_{y, z}^1$ is equal to that of the (y, T, z) -Markov bridge. In particular, it holds that

$$\mathbb{P}_{y, z}^1(Y_0 = y, Y_{T-} = z) = 1.$$

The corresponding transition densities that satisfy

$$p^{(z, T)}(s, y; t, y') m(dy') := \mathbb{P}_{y, z}^1(Y_t \in dy' | Y_s = y)$$

for $y, y', z \in E$ and $0 < s < t < T$ are expressed by

$$p^{(z, T)}(s, y; t, y') = \frac{\tilde{p}_{t-s}(y, y') \tilde{p}_{T-t}(y', z)}{\tilde{p}_{T-s}(y, z)}.$$

Furthermore, we introduce another probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ and on which we consider the random variable X with law $\nu := \mathbb{P}_2 \circ X^{-1}$. Let $\Omega := \Omega_T^1 \times \Omega_2$, $\mathcal{F}_t := \mathcal{F}_t^1 \otimes \mathcal{F}_2$, $\mathcal{F}_{T-} = \mathcal{F}_{T-}^1 \otimes \mathcal{F}_2$, and let $\mathbb{P}_y := \tilde{\mathbb{P}}_y^1 \otimes \mathbb{P}_2$ such that

$$d\mathbb{P}_y(\omega_1, \omega_2) \big|_{\mathcal{F}_t} := \Lambda_t^{(y, T, X)}(\omega_1, \omega_2) d\tilde{\mathbb{P}}_y^1(\omega_1) \otimes d\mathbb{P}_2(\omega_2) \big|_{\mathcal{F}_t} \quad t \in [0, T) \quad (2.1)$$

is satisfied, where

$$\Lambda_t^{(y, T, X)}(\omega_1, \omega_2) := \frac{\tilde{p}_{T-t}(Y_t(\omega_1), X(\omega_2))}{\tilde{p}_T(y, X(\omega_2))}. \quad (2.2)$$

Then, on the filtered product space $(\Omega, \mathcal{F}_{T-}, \mathbb{P}_y, (\mathcal{F}_t)_{t \in [0, T)})$, the RMB $(Z_t)_{t \in [0, T]}$ is obtained by setting $Z_t(\omega_1, \omega_2) := Y_t(\omega_1)$ for $t \in [0, T)$ and $Z_T(\omega_1, \omega_2) := X(\omega_2)$. For the filtering problem, we have the following:

Proposition 2.1. *For $t \in [0, T)$, the conditional probability (or the Bayesian posterior probability) (1.2) is given by*

$$\pi_t(dz) = \frac{\frac{\tilde{p}_{T-t}(Z_t, z)}{\tilde{p}_T(Z_0, z)} \nu(dz)}{\int_E \frac{\tilde{p}_{T-t}(Z_t, z')}{\tilde{p}_T(Z_0, z')} \nu(dz')}.$$

Remark 2.1. From a stochastic filtering viewpoint, it is interesting that the conditional probability (or the Bayesian posterior probability) $\pi_t(dz)$ is expressed by a function of t , Z_t , Z_0 , and ν . At each point in time, $\pi_t(dz)$ can be computed by inserting the current observation Z_t and the initial observation Z_0 , only. The past information (memory) $(Z_s)_{s \in (0, t)}$ is not necessary.

We note that once the initial value Z_0 is fixed, the process $(Z_t)_{t \in [0, T]}$ is “Markovian” in the following sense:

Proposition 2.2. *For $0 \leq s < t < T$, (1.4) holds where*

$$P_{s,t}(x, dy|z_0) = q(s, x; t, y|z_0)m(dy)$$

and

$$q(s, x; t, y|z_0) := \frac{\tilde{p}_{t-s}(x, y) \left\{ \int_E \frac{\tilde{p}_{T-t}(y, z)}{\tilde{p}_T(z_0, z)} \nu(dz) \right\}}{\int_E \frac{\tilde{p}_{T-s}(x, z')}{\tilde{p}_T(z_0, z')} \nu(dz')}.$$

For a fixed z_0 , the transition density $q(s, x; t, y|z_0)$ satisfies the Chapman-Kolmogorov identity:

$$q(s, x; u, z|z_0) = \int_E q(s, x; t, y|z_0) q(t, y; u, z|z_0) m(dy)$$

for $0 < s < t < u < T$ and $x, y, z \in E$.

Remark 2.2. Let $m(dx) = dx$ and $\nu(dx) = g(x)dx$. Then, one can check that (a)

$$\lim_{t \uparrow T} \int_E \frac{\tilde{p}_{T-t}(y, z)}{\tilde{p}_T(Z_0, z)} \nu(dz) = \int_E \frac{\delta_y(z)}{\tilde{p}_T(Z_0, z)} g(z) dz = \frac{g(y)}{\tilde{p}_T(Z_0, y)}$$

where $\delta_y(\cdot)$ is the Dirac delta function with point mass at y , and (b)

$$\lim_{t \uparrow T} P_{s,t}(Z_s, dx|Z_0) = \frac{\frac{\tilde{p}_{T-s}(Z_s, y)}{\tilde{p}_T(Z_0, y)} \nu(dy)}{\int_E \frac{\tilde{p}_{T-s}(Z_s, z')}{\tilde{p}_T(Z_0, z')} \nu(dz')} = \pi_s(dx).$$

Next, let us suppose that the underlying (unconditioned) Markov process is the solution to the Markovian SDE,

$$dY_t = b(Y_t)dt + \sigma(Y_t)d\tilde{W}_t, \quad Y_0 = y \quad (2.3)$$

on $(\Omega_T^1, \mathcal{F}_T^1, \tilde{\mathbb{P}}_y^1, (\mathcal{F}_t^1)_{t \in [0, T]})$, where $\tilde{W} := (\tilde{W}_t)_{t \in [0, T]}$ is a standard n -dimensional $(\tilde{\mathbb{P}}_y^1, \mathcal{F}_t^1)$ -Brownian motion, $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$. We assume that there exists a unique strong solution to (2.3) and that there exist the associated transition densities $\tilde{p}_t(x, y) > 0$ for $t \in (0, T]$, $x, y \in E$, which are sufficiently smooth with respect to (t, x) . We recall that $\tilde{p}_{(\cdot)}(\cdot, y)$ satisfies the Kolmogorov backward equation

$$\begin{aligned} (-\partial_t + \mathcal{L}) \tilde{p}_t(x, y) &= 0 \quad \text{for } (t, x) \in (0, T] \times E, \\ \tilde{p}_0(x, y) &= \delta_y(x), \end{aligned} \quad (2.4)$$

where

$$\mathcal{L}(\cdot) := b(x)^\top \nabla(\cdot) + \frac{1}{2} \text{tr}((\sigma \sigma^\top)(x) \nabla \nabla(\cdot))$$

is the infinitesimal generator of the diffusion Y . In this case, by using Itô's formula, we can see that

$$\begin{aligned} \frac{d\mathbb{P}_{y,z}^1}{d\mathbb{P}_y^1} \Big|_{\mathcal{F}_t^1} &= \Lambda_t^{(y,T,z)} = \exp \left[\int_0^t \nabla \log \tilde{p}_{T-u}(Y_u, z)^\top \sigma(Y_u) d\tilde{W}_u \right. \\ &\quad \left. - \frac{1}{2} \int_0^t |\sigma(Y_u)^\top \nabla \log \tilde{p}_{T-u}(Y_u, z)|^2 du \right] \end{aligned}$$

for $t \in [0, T)$. The SDE defined on $(\Omega, \mathcal{F}_{T-}, \mathbb{P}_y, (\mathcal{F}_t)_{t \in [0, T)})$ and satisfied by $Z := (Z_t)_{t \in [0, T)}$ can now be written down as

$$dZ_t = \{b(Z_t) + (\sigma\sigma^\top)(Z_t) \nabla \log \tilde{p}_{T-t}(Z_t, X)\} dt + \sigma(Z_t) dW_t \quad (Z_0 = y),$$

where $W := (W_t)_{t \in [0, T)}$ is a $(\mathbb{P}_y, \mathcal{F}_t)$ -Brownian motion. Further, we deduce that $Z_{T-} := \lim_{t \uparrow T} Z_t = X$ \mathbb{P}_y -a.s. We introduce the notation

$$\rho_t(f) := \int_E f(z) \rho_t(dz)$$

for the expectation expressed in terms of the so-called “unnormalised conditional probability” $\rho_t(dz)$ given by

$$\rho_t(dz) := \frac{\tilde{p}_{T-t}(Z_t, z)}{\tilde{p}_T(y, z)} \nu(dz).$$

Then, we see that, for a Borel measurable function f ,

$$\pi_t(f) = \frac{\rho_t(f)}{\rho_t(1)}.$$

Proposition 2.3. *Define*

$$\ell_s(z) := \log \tilde{p}_{T-s}(Z_s, z), \quad z \in E,$$

and suppose that, for any $t \in [0, T)$,

$$\int_E \frac{f(z)}{\tilde{p}_T(Z_0, z)} \left[\int_0^t |\sigma^\top(Z_s) \tilde{p}_{T-s}(Z_s, z) \nabla \ell_s(z)|^2 ds \right]^{1/2} \nu(dz) < \infty, \quad (2.5)$$

where $y = Z_0 \in E$. Then, the following statements are valid:

(1) $(\rho_t(f))_{t \in [0, T)}$ solves the Zakai equation

$$\rho_t(f) = \rho_0(f) + \int_0^t \rho_s(f \nabla \ell_s)^\top \{dZ_s - b(Z_s) ds\}.$$

(2) $(\pi_t(f))_{t \in [0, T)}$ solves the Kushner-Stratonovich equation

$$\begin{aligned} \pi_t(f) &= \pi_0(f) \\ &+ \int_0^t \{\pi_s(f \nabla \ell_s) - \pi_s(f) \pi_s(\nabla \ell_s)\}^\top [dZ_s - \{b(Z_s) + (\sigma\sigma^\top)(Z_s) \pi_s(\nabla \ell_s)\} ds]. \end{aligned} \quad (2.6)$$

Remark 2.3. It follows that, for $t \in [0, T)$, the process

$$M_t := Z_t - \int_0^t \{b(Z_s) + (\sigma\sigma^\top)(Z_s)\pi_s(\nabla\ell_s)\} ds$$

is a $(\mathbb{P}_y, \mathcal{F}_t^Z)$ -martingale, if the \mathcal{F}_t -local martingale $\int_0^{(\cdot)} \sigma(Z_u) dW_u$ satisfies

$$\mathbb{E}_y \left[\sup_{0 \leq t \leq T} \left| \int_0^t \sigma(Z_u) dW_u \right| \right] < \infty.$$

Indeed, M is \mathcal{F}_t^Z -adapted, and satisfies, for $0 \leq s \leq t < T$,

$$\begin{aligned} & \mathbb{E}_y[M_t - M_s | \mathcal{F}_s^Z] \\ &= \mathbb{E}_y \left[Z_t - Z_s - \int_s^t \{b(Z_u) + (\sigma\sigma^\top)(Z_u) \mathbb{E}[\nabla\ell_u(X) | \mathcal{F}_u^Z]\} du \mid \mathcal{F}_s^Z \right] \\ &= \mathbb{E}_y \left[\int_s^t \sigma(Z_u) dW_u + \int_s^t (\sigma\sigma^\top)(Z_u) \{ \nabla\ell_u(X) - \mathbb{E}[\nabla\ell_u(X) | \mathcal{F}_u^Z] \} du \mid \mathcal{F}_s^Z \right] \\ &= \mathbb{E}_y \left[\mathbb{E}_y \left[\int_s^t \sigma(Z_u) dW_u \mid \mathcal{F}_s \right] \mid \mathcal{F}_s^Z \right] \\ &+ \mathbb{E}_y \left[\int_s^t \mathbb{E}_y [(\sigma\sigma^\top)(Z_u) \{ \nabla\ell_u(X) - \mathbb{E}[\nabla\ell_u(X) | \mathcal{F}_u^Z] \} \mid \mathcal{F}_u^Z] du \mid \mathcal{F}_s^Z \right] \\ &= 0. \end{aligned}$$

Here we use the inclusions $\mathcal{F}_s^Z \subset \mathcal{F}_s$ and $\mathcal{F}_s^Z \subset \mathcal{F}_u^Z$ for $0 \leq s \leq u < T$ and the tower property of conditional expectation. By making use of the martingale M , we rewrite (2.6) in the differential form as

$$d\pi_t(f) = \{\pi_t(f\nabla\ell_t) - \pi_t(f)\pi_t(\nabla\ell_t)\}^\top dM_t,$$

or

$$d\pi_t(f) = \{\pi_t(f\nabla\ell_t) - \pi_t(f)\pi_t(\nabla\ell_t)\}^\top (\sigma\sigma^\top)^{1/2}(Z_t) dB_t.$$

The n -dimensional $(\mathbb{P}_y, \mathcal{F}_t^Z)$ -Brownian motion $(B_t)_{t \in [0, T]}$ is defined by

$$B_t := \int_0^t (\sigma\sigma^\top)^{-1/2}(Z_u) dM_u,$$

where it is assumed that $\sigma\sigma^\top(z) > 0$ for all $z \in E$. In financial applications, this type of SDEs are applied as models for the dynamics of discounted asset prices under the risk-neutral probability measure.

3 Proofs

In this section we collect the proofs to the propositions at the basis of the randomised Markov bridges.

3.1 Proof of Proposition 2.1

By applying the Bayes rule and the relations (2.1) and (2.2), we have

$$\pi_t(f) := \mathbb{E}_y [f(X) | \mathcal{F}_t^Z] = \frac{\tilde{\mathbb{E}}_y [\Lambda_t^{y,T,X} f(X) | \mathcal{F}_t^Z]}{\tilde{\mathbb{E}}_y [\Lambda_t^{y,T,X} | \mathcal{F}_t^Z]},$$

where we use the notation $\tilde{\mathbb{E}}_y[\cdot]$ for the expectation with respect to the probability measure $d\tilde{\mathbb{P}}_y^1 \otimes d\mathbb{P}_2$. Observing that \mathcal{F}_t^Z and X are independent under $d\tilde{\mathbb{P}}_y^1 \otimes d\mathbb{P}_2$, we deduce that

$$\pi_t(f) = \frac{\int_E \frac{\tilde{p}_{T-t}(Z_t, z)}{\tilde{p}_T(y, z)} f(z) \nu(dz)}{\int_E \frac{\tilde{p}_{T-t}(Z_t, z')}{\tilde{p}_T(y, z')} \nu(dz')}.$$

□

3.2 Proof of Proposition 2.2

The proof is similar to the one for Proposition 2.1. We see that, for $0 \leq s \leq t < T$,

$$\mathbb{E}_{z_0} [f(Z_t) | \mathcal{F}_s^Z] = \frac{\tilde{\mathbb{E}}_{z_0} [\Lambda_t^{y,T,X} f(Z_t) | \mathcal{F}_s^Z]}{\tilde{\mathbb{E}}_{z_0} [\Lambda_t^{y,T,X} | \mathcal{F}_s^Z]},$$

where we use the Bayes rule, (2.1) and (2.2). Recalling that \mathcal{F}_t^Z and X are independent under $d\tilde{\mathbb{P}}_y^1 \otimes d\mathbb{P}_2$, the numerator of the right-hand-side of the above equation is equal to

$$\int_E \tilde{p}_{t-s}(z_0, y) \left\{ \int_E \frac{\tilde{p}_{T-t}(y, z)}{\tilde{p}_T(z_0, z)} \nu(dz) \right\} f(y) m(dy).$$

By computing the denominator in a similar way, we obtain the expression for $P_{s,t}(x, dy | z_0)$. The Chapman-Kolmogorov identity is seen from

$$\mathbb{E}[f(Z_u) | \mathcal{F}_s^Z] = \mathbb{E} [\mathbb{E} [f(Z_u) | \mathcal{F}_t^Z] | \mathcal{F}_s^Z], \quad 0 \leq s \leq t \leq u < T,$$

which is the tower property of conditional expectation. □

3.3 Proof of Proposition 2.3

By the Kolmogorov backward equation (2.4) and Itô's formula, we have

$$\begin{aligned}\tilde{p}_{T-t}(Z_t, z) &= \tilde{p}_T(Z_0, z) + \int_0^t \nabla \tilde{p}_{T-s}(Z_s, z)^\top dZ_s \\ &\quad + \int_0^t \left\{ -\partial_s \tilde{p}_{T-s}(Z_s, z) + \frac{1}{2} \text{tr}((\sigma \sigma^\top)(Z_s) \nabla \nabla \tilde{p}_{T-s}(Z_s, z)) \right\} ds \\ &= \tilde{p}_T(Z_0, z) + \int_0^t \tilde{p}_{T-s}(Z_s, z) \nabla \log \tilde{p}_{T-s}(Z_s, z)^\top \{dZ_s - b(Z_s) ds\}.\end{aligned}$$

Then, by making use of this relation, we deduce that

$$\begin{aligned}\rho_t(f) &= \int_E f(z) \frac{\tilde{p}_{T-t}(Z_t, z)}{\tilde{p}_T(Z_0, z)} \nu(dz) \\ &= \rho_0(f) + \int_E \frac{f(z)}{\tilde{p}_T(Z_0, z)} \left[\int_0^t \tilde{p}_{T-s}(Z_s, z) \nabla \log \tilde{p}_{T-s}(Z_s, z)^\top \{dZ_s - b(Z_s) ds\} \right] \nu(dz) \\ &= \rho_0(f) + \int_0^t \left[\int_E f(z) \nabla \log \tilde{p}_{T-s}(Z_s, z)^\top \frac{\tilde{p}_{T-s}(Z_s, z)}{\tilde{p}_T(Z_0, z)} \nu(dz) \right] \{dZ_s - b(Z_s) ds\} \\ &= \rho_0(f) + \int_0^t \rho_s(f \nabla \ell_s^\top) \{dZ_s - b(Z_s) ds\}.\end{aligned}$$

Here the stochastic version of Fubini's theorem is employed while recalling the condition (2.5); we refer to Theorem 2.2 in Veraar (2012). Therefore, an application of Itô's formula one can show that

$$\begin{aligned}d\pi_t(f) &= \frac{d\rho_t(f)}{\rho_t(1)} - \frac{\rho_t(f) d\rho_t(1)}{\rho_t(1)^2} - \frac{d\langle \rho(f), \rho(1) \rangle_t}{\rho_t(1)^2} + \frac{\rho(f) d\langle \rho(1) \rangle_t}{\rho_t(1)^3} \\ &= \frac{\rho_t(f \nabla \ell_t^\top) \{dZ_t - b(Z_t) dt\}}{\rho_t(1)} - \frac{\rho_t(f) \rho_t(\nabla \ell_t^\top) \{dZ_t - b(Z_t) dt\}}{\rho_t(1)^2} \\ &\quad - \frac{\rho_t(f \nabla \ell_t^\top) (\sigma \sigma^\top)(Z_t) \rho_t(\nabla \ell_t)}{\rho_t(1)^2} dt + \frac{\rho_t(f) \rho_t(\nabla \ell_t^\top) (\sigma \sigma^\top)(Z_t) \rho_t(\nabla \ell_t)}{\rho_t(1)^3} dt \\ &= \{ \pi_t(f \nabla \ell_t^\top) - \pi_t(f) \pi_t(\nabla \ell_t^\top) \} [dZ_t - \{b(Z_t) + (\sigma \sigma^\top)(Z_t) \pi_t(\nabla \ell_t)\} dt]. \quad \square\end{aligned}$$

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References

- [1] BAUDOIN, F. (2002). Conditioned stochastic differential equations: theory, examples and application to finance. *Stochastic Processes and their Applications*, **100**, 109–145.
- [2] BRODY, D. C., L. P. HUGHSTON AND A. MACRINA (2008). Information-based asset pricing. *Int. J. Theor. Appl. Finance*, **11** (1), 107–142.
- [3] FILIPOVIĆ, D., L. P. HUGHSTON AND A. MACRINA (2012). Conditional density models for asset pricing. *Int. J. Theor. Appl. Finance*, **15** (1), 1250002, 24 pp.
- [4] FITZSIMMONS, P., J. PITMAN, AND M. YOR (1993). Markovian bridges: construction, palm interpretation, and splicing. *Seminar on Stochastic Processes, 1992. (Progress in Probability, Volume 33, Editors: E. Çinlar, K. L. Chung, M. J. Sharpe, R. F. Bass, and K. Burdzy)*, 101–134, Birkhäuser.
- [5] HOYLE, E., L. P. HUGHSTON AND A. MACRINA (2011). Lévy random bridges and the modelling of financial information. *Stochastic Processes and their Applications*, **121**, 856–884.
- [6] VERAAR, M. (2012). The stochastic Fubini theorem revisited. *Stochastics*, **84**(4), 543–551.